ST402 Principals and Methods of Statistical Practice
Mid-term Test Solutions

1. (a) The probabilities here are easily found

$$
\begin{aligned}
P(X=1) &= \frac{2}{3} \\
P(X=2) &= \frac{1}{3} \\
P(Y=0|X=1) &= \frac{1}{3} \\
P(Y=1|X=1) &= \frac{2}{3} \\
P(Y=1|X=2) &= \frac{2}{3} \\
P(Y=2|X=2) &= \frac{1}{3}.
\end{aligned}
$$

So

$$
\begin{aligned}
E(Y|X=1) &= 0 \times \frac{1}{3} + 1 \times \frac{2}{3} = \frac{2}{3} \\
E(Y|X=2) &= 1 \times \frac{2}{3} + 2 \times \frac{1}{3} = \frac{4}{3},
\end{aligned}
$$

and

$$
\begin{gathered}
E(Y) = E(Y|X=1)P(X=1) + E(Y|X=2)P(X=2) \\
= \frac{2}{3} \times \frac{2}{3} + \frac{4}{3} \times \frac{1}{3} = \frac{8}{9}.
\end{gathered}
$$

(b)

$$
\begin{aligned}
P(Y=0) &= P(Y=0|X=1)P(X=1) = \frac{1}{3} \times \frac{2}{3} = \frac{2}{9}. \\
P(Y=1) &= P(Y=1|X=1)P(X=1) + P(Y=1|X=2)P(X=2) \\
&= \frac{2}{3} \times \frac{2}{3} + \frac{2}{3} \times \frac{1}{3} = \frac{6}{9}. \\
P(Y=2) &= P(Y=2|X=2)P(X=2) = \frac{1}{3} \times \frac{1}{3} = \frac{1}{9}.
\end{aligned}
$$

2. (a) The total probability is 1, so

$$
\begin{aligned}
1 &= \int_0^1 \left(\int_0^{1-x} k(x+y)dy \right) dx \\
&= \int_0^1 \left[\frac{k(x+y)^2}{2} \right]_{y=0}^{y=1-x} dx \\
&= \int_0^1 \frac{k}{2}(1-x^2)dx = \frac{k}{2}\left[x - \frac{x^3}{3} \right]_0^1 = \frac{k}{3}.
\end{aligned}
$$

So, $k = 3$.

(b)

$$\begin{aligned} f_Y(y) &= \int_0^{1-y} k(x+y)dx \\ &= \left[\frac{3(x+y)^2}{2}\right]_{x=0}^{x=1-y} \\ &= \frac{3}{2}(1-y^2). \end{aligned}$$

If $0 < y < 1$, $0 < x < 1 - y$ then

$$\begin{aligned} f_{X|Y}(x|y) &= \frac{f_{X,Y}(x,y)}{f_Y(y)} \\ &= \frac{3(x+y)}{\frac{3}{2}(1-y^2)} \\ &= \frac{2(x+y)}{1-y^2}. \end{aligned}$$

For other values of x, y, we have $f_{X|Y}(x|y) = 0$.

$$\begin{aligned} E\left[X \middle| Y = \frac{1}{2}\right] &= \int_0^{\frac{1}{2}} x\frac{2(\frac{1}{2}+x)}{1-\frac{1}{4}}dx \\ &= \frac{4}{3}\int_0^{\frac{1}{2}} x(1+2x)dx \\ &= \frac{4}{3}\left[\frac{x^2}{2}+\frac{2x^3}{3}\right]_0^{\frac{1}{2}} \\ &= \frac{4}{3}\left[\frac{1}{8}+\frac{2\times\frac{1}{8}}{3}\right] \\ &= \frac{5}{18}. \end{aligned}$$

3. (a) Because of independence

$$f_{Y,Z}(y,z) = \begin{cases} \lambda^2 e^{-\lambda(y+z)} & 0 < y < \infty,\ 0 < z < \infty \\ 0 & \text{elsewhere} \end{cases}.$$

(b)

$$\begin{aligned} u &= z + y \\ v &= \frac{z}{y} \end{aligned}$$

so

$$\begin{aligned} y &= \frac{u}{1+v} \\ z &= \frac{uv}{1+v}. \end{aligned}$$

$$
\begin{aligned}
\frac{\partial(u,v)}{\partial(y,z)} &= \begin{vmatrix} \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \end{vmatrix} \\
&= \begin{vmatrix} 1 & 1 \\ -\frac{z}{y^2} & \frac{1}{y} \end{vmatrix} \\
&= \frac{1}{y} + \frac{z}{y^2} \\
&= \frac{y+z}{y^2} \\
&= \frac{(1+v)^2}{u}.
\end{aligned}
$$

So, for $0 < u < \infty$, and $0 < v < \infty$,

$$
\left|\frac{\partial(u,v)}{\partial(y,z)}\right| = \frac{(1+v)^2}{u}
$$

(c)

$$
\begin{aligned}
f_{U,V}(u,v) &= \lambda^2 e^{-\lambda u} \left|\frac{\partial(u,v)}{\partial(y,z)}\right| \\
&= \begin{cases} \lambda^2 e^{-\lambda u} \frac{u}{(1+v)^2} & 0 < u < \infty,\ 0 < y < \infty \\ 0 & \text{elsewhere} \end{cases}.
\end{aligned}
$$

Since the joint density factorises into the product of a gamma density function for U and a Pareto density function for V, the random variables U, V are independent

4. Let

(a)

$$
\begin{aligned}
E[X] &= E[E[X|N]] \\
&= E\left[E\left[\sum_{i=1}^{N} Y_i \,|N\right]\right] \\
&= E\left[\sum_{i=1}^{N} E\left[Y_i \,|N\right]\right] \\
&= E\left[N\frac{1}{2}\right] \\
&= \frac{\lambda}{2}.
\end{aligned}
$$

As for the MGF,

$$\begin{aligned}
E\left[e^{tX}\right] &= E\left[E\left[e^{tX}\,|N\right]\right] \\
&= E\left[E\left[e^{t\sum_{i=1}^{N} Y_i}\,|N\right]\right] \\
&= E\left[\prod_{i=1}^{N} E\left[e^{tY_i}\,|N\right]\right] \\
&= E\left[M_Y(t)^N\right] \\
&= E\left[\left(\frac{1}{2}(1+e^t)\right)^N\right] \\
&= \exp\left[\lambda\left[\frac{1}{2}\left(1+e^t\right)-1\right]\right] \\
&= \exp\left[\frac{\lambda}{2}\left(e^t-1\right)\right].
\end{aligned}$$

So X has a Poisson distribution with mean $\lambda/2$.

(b) Now we have

$$M_N(s) = \frac{\frac{1}{2}}{1-\frac{1}{2}e^s} = \frac{1}{2-e^s}.$$

So,

$$E[X] = E[N]E[Y] = 1 \times \frac{1}{2} = \frac{1}{2},$$

and

$$\begin{aligned}
M_X(t) &= \frac{1}{2-\frac{1}{2}(1+e^t)} \\
&= \frac{1}{\frac{3}{2}-\frac{1}{2}e^t} \\
&= \frac{\frac{2}{3}}{1-\frac{1}{3}e^t}.
\end{aligned}$$

So, X has a geometric distribution with

$$P(X=k) = \frac{2}{3}\left(\frac{1}{3}\right)^k$$

for $k = 0, 1, 2, \ldots$.